FINAL: ALGEBRA II

Date: **29th April 2024**
The Total points is **111** and the maximum you can score is **100** points.
Notation: $\mathbb{F}_p$ denotes the finite field of p elements where p is a prime number.

(1) (8+8=16) Mark all correct options.
  (a) Let $f(x) \in \mathbb{Q}(x)$ be an irreducible degree 4 polynomial. Which of the following can be a Galois group of $f(x)$?
    (i) $\mathbb{Z}/4\mathbb{Z}$
    (ii) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
    (iii) The alternating group A_4
    (iv) The symmetric group S_4
  (b) Which of the following statements are true?
    (i) Every finite extension of $\mathbb{Q}(x)$ is separable.
    (ii) Every finite extension of $\mathbb{F}_p(x)$ is separable. Here x is an indeterminate.
    (iii) Every inseparable extension of $\mathbb{F}_p(x)$ is finite.
    (iv) Every finite Galois extension of $\mathbb{F}_p(x)$ is separable. Here x is an indeterminate.

(2) (15+15=30 points) Prove or disprove.
  (a) The field $\mathbb{Q}(\sqrt[3]{5})$ is a subfield of some cyclotomic field over $\mathbb{Q}$.
  (b) Every finite extension of a finite field is Galois.

(3) (15+15=30 points) Let F be a field. Let $f(x) = f_1(x)f_2(x) \in F[x]$ be a factorization into irreducible factors and assume $f(x)$ is separable. Let K be the splitting field of f. Let K_1 and K_2, contained in K, be the splitting fields of f_1 and f_2 respectively. Let G_1 and G_2 be the Galois group of K_1/F and K_2/F respectively. Assume that $f_2(x)$ is irreducible in $K_1[x]$. Show by an example that the Galois group of K/F need not be $G_1 \times G_2$. Let $\alpha \in K$ be a root of $f_2(x)$. If $K_2 = F(\alpha)$, show that Galois group of K/F is $G_1 \times G_2$.

(4) (10+10=20 points) Let a real number a be constructible by straight edge and compass. Show that the Galois group of the Galois closure of $\mathbb{Q}(a)/\mathbb{Q}$ is a solvable group.
Let $K/\mathbb{Q}$ be a Galois extension with Galois group A_5. Let $a \in K \setminus \mathbb{Q}$ be a real number, show that a is not constructible by straight edge and compass.

(5) (5+10=15 points) Let F be a field of characteristic $p > 0$. When is a finite field extension K/F called purely inseparable? Show that such a field extension is a normal extension.